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Nonsimilar solutions for mixed convection in non-Newtonian fluids along a wedge with variable surface temperature in a porous medium

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Abstract *A nonsimilar boundary layer analysis is presented for the problem of mixed convection in power-law type non-Newtonian fluids along a vertical wedge with variable wall temperature distribution. The mixed convection regime is divided into two regions, namely, the forced convection dominated regime and the free convection dominated regime. The two solutions are matched. Numerical results are presented for the details of the velocity and temperature fields. A discussion is provided for the effect of viscosity index on the surface heat transfer rate.*

Nomenclature

f = dimensionless stream function
g = acceleration due to gravity
h = heat transfer coefficient
k = thermal conductivity
K = permeability for the porous medium
L = plate length
m = wedge flow parameter
n = viscosity index
Nu = Nusselt number
Pe = Peclet number
q_w = wall heat flux
Ra = Rayleigh number
T = Temperature
u,v = velocity components in x and y directions
U_∞ = free stream velocity
x,y = axial and normal coordinates
α = effective thermal diffusivity of porous medium
β = volumetric coefficient of thermal expansion
γ = half-wedge angle
η = similarity variable
θ = dimensionless temperature
ν = kinematic viscosity
ξ = nonsimilar parameter
ρ = density of fluid
μ = consistency index for viscosity
τ_w = wall shear stress
ψ = stream function

Subscripts

w = wall conditions
∞ = free stream conditions

Introduction

The motivation for the problem studied in this paper was the numerous thermal engineering applications such as geothermal systems, crude oil extraction,

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thermal insulation and ground water pollution. Cheng and Minkowycz (1977, pp. 2040-9) presented similarity solutions for free convective heat transfer from a vertical plate in a fluid-saturated porous medium. Gorla and co-workers (Gorla and Zinolabedini, 1987, pp. 26-30; Gorla and Tornabene, 1988, pp. 95-106) solved the nonsimilar problem of free convective heat transfer from a vertical plate embedded in a saturated porous medium with an arbitrarily varying surface temperature or heat flux. The problem of combined convection from vertical plates in porous media was studied by Minkowycz *et al.* (1985, pp. 349-59) and Ranganathan and Viskanta (1984, pp. 305-17). Nakayama and Pop (1985, pp. 683-97) presented similarity solutions for the free, forced and combined convection. Hsieh *et al.* (1993, pp. 1485-93) derived nonsimilar solutions for combined convection from vertical plates in porous media. Kumari and Gorla (1997, pp. 393-8) examined the combined convection along a non-isothermal wedge in a porous medium. All these studies were concerned with Newtonian fluid flows. A number of industrially important fluids including fossil fuels which may saturate underground beds display non-Newtonian behavior. Non-Newtonian fluids exhibit a nonlinear relationship between shear stress and shear rate.

Chen and Chen (1988, pp. 257-60) presented similarity solutions for free convection of non-Newtonian fluids over vertical surfaces in porous media. Nakayama and Koyama (1991, pp. 55-70) studied the natural convection over a non-isothermal body of arbitrary shape embedded in a porous medium. Gorla and co-workers (Kumari *et al.*, 1997, pp. 34-7; Gorla and Kumari, 1999, 1996, pp. 55-64; Gorla *et al.*, 1997a, pp. 319-34, 1997b, pp. 281-6; Kumari and Gorla, 1996, pp. 157-66; Gorla and Takhar, 1997, pp. 596-608) have recently analyzed the problems of mixed convection in non-Newtonian fluids along vertical and horizontal plates in porous media.

The present work has been undertaken in order to analyze the mixed convection from a vertical non-isothermal wedge embedded in non-Newtonian fluid saturated porous media. The boundary condition of variable surface temperature is treated in this paper. The power law model of Ostwald-de-Waele, which is adequate for many non-Newtonian fluids, is considered here. The governing equations are first transformed into a dimensionless form and the resulting nonsimilar set of equations is solved by a finite difference method. Numerical results for the velocity and temperature fields are presented.

Analysis

Let us consider the mixed convention in a porous medium from an impermeable wedge, which is heated and has a variable wall temperature. The properties of the fluid and the porous medium are assumed to be constant and isotropic. The Darcy model is considered which is valid under conditions of small pores of porous medium and flow velocity. Also, the slip velocity at the wall is imposed, which has a smaller effect on the heat transfer results as the distance from the leading edge increases. The axial and normal coordinates are x and y, and the corresponding flow velocities are u and v respectively. Figure 1 shows the coordinate system and model of the flow. The gravitational

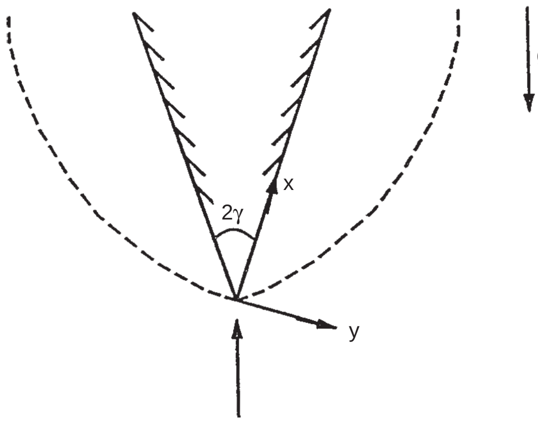

Figure 1.
Coordinate system and flow model

acceleration g is acting downwards opposite to the normal coordinate y. The governing equations under the Boussinesq and boundary layer approximations are given by,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u^n = U_\infty^n + \frac{K}{\mu}\rho g_x \beta (T - T_\infty) \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

where $g_x = gCos(\gamma)$

In the above equations, T is the temperature of the wall, n is the viscosity index, ρ is the density, K is the permeability of porous medium, β is the volumetric coefficient of thermal expansion, μ is the viscosity, α is the equivalent thermal diffusivity of the porous medium. With power law variation in wall temperature, the boundary conditions can be written as

$$\begin{aligned} y = 0 &: \quad v = 0, \quad (T - T_\infty) = Ax^\lambda \\ y = \infty &: \quad u = U_\infty, \quad T = T_\infty \end{aligned} \tag{4}$$

where A and λ are prescribed constants. Note that $\lambda = 0$ corresponds to the case of uniform wall temperature.

A. Forced convection dominated regime

The continuity equation is automatically satisfied by defining a stream function ψ (x,y) such that

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x}$$

Proceeding with the analysis, we define the following transformations:

$$\eta = \frac{y}{x} \ Pe_x^{1/2}$$
$$\psi = \alpha \, Pe_x^{1/2} \, f(\xi_f, \ \eta)$$
$$\xi_f = \left(\frac{Ra_x}{Pe_x}\right)^n$$
$$\theta = \frac{T - T_\infty}{T_w - T_\infty} \tag{5}$$
$$Pe_x = \frac{U_\infty \, x}{\alpha}$$
$$Ra_x = \frac{x}{\alpha} \ \left(\frac{\rho K g_x \beta \Delta T_w}{\mu}\right)^{1/n}$$
$$U_\infty = cx^m$$
$$m = \gamma/(\pi - \gamma)$$

The governing equations and boundary conditions, equations (1) - (4), can then be transformed into

$$(f')^n = 1 + \xi\theta \tag{6}$$

$$\theta'' - \lambda f' \theta + \left(\frac{m+1}{2}\right) f \theta' = (\lambda - m) \, \xi_f \left[f' \frac{\partial\theta}{\partial\xi_f} - \theta' \frac{\partial f}{\partial\xi_f}\right. \tag{7}$$

$$\left(\frac{m+1}{2}\right) f \, (\xi_f, \ 0) + (\lambda - m) \, \xi_f \frac{\partial f}{\partial\xi_f} \ (\xi_f, \ 0) = 0$$
$$\text{or} \quad f(\xi_f, \ 0) = 0, \theta(\xi_f, \ 0) = 1, f'(\xi_f, \ \infty) = 1, \theta(\xi_f, \ \infty) = 0 \tag{8}$$

The primes in the above equations denote partial differentiations with respect to η.

In the above system of equations, the dimensionless parameter ξ_f is a measure of the buoyancy effect on forced convection. The case of $\xi_f = 0$ corresponds to pure forced convection. The limiting case of $\xi_f = \infty$ corresponds to pure free convection region. The equations (6) - (8) cannot be solved for the entire regime of mixed convection because of singularity at $\xi_f = \infty$. The above system of equations is used to solve the region covered by $\xi_f = 0 - 1$ to provide the first half of the total solution of the mixed convection regime.

Some of the physical quantities of interest include the velocity components u and v in the x and y directions and the local Nusselt number $Nu_x = hx/k$ where $h = q_w/[T_w(x) - T_\infty]$. They are given by

$$u = U_\infty \, f'(\xi_f, \eta) \tag{9}$$

$$v=-\left(\frac{\alpha}{x}\right)Pe_x^{1/2}\left(\left(\frac{m+1}{2}\right)f(\xi_f,\eta)\right.$$

$$\left.-\left(\frac{m-1}{2}\right)\eta f'(\xi_f,\eta)+(\lambda-m)\xi\frac{\partial f}{\partial\xi}\right) \quad (10)$$

$$Nu_x=-Pe_x^{1/2}\,\theta(\xi_{f,}0) \quad (11)$$

B. Free convection dominated regime

For buoyancy dominated regime the following dimensionless variables are introduced in the transformation

$$\eta = \frac{y}{x}(Ra_x)^{\frac{1}{2}}\ \xi_n=\left(\frac{Pe_x}{Ra_x}\right)^n \quad (12)$$

$$\psi = \alpha\ (Ra_x)^{\frac{1}{2}}\ f(\xi_{n,}\ \eta)\ \theta(\xi_{n,}\ \eta)\ =\ \left(\frac{T-T_\infty}{T_w(x)-T_\infty}\right) \quad (13)$$

Substituting equations (12) and (13) into the governing equations (1) - (4) leads to

$$(f')^n=\xi+\theta \quad (14)$$

$$\theta''+\frac{\lambda\ +n}{2n}\ f\ \theta'-\lambda\ f'\ \theta\ =\ (m-\lambda)\ \xi_n\left[f'\ \frac{\partial\theta}{\partial\xi_n}\ -\ \theta'\ \frac{\partial f}{\partial\xi_n}\right] \quad (15)$$

$$\frac{(\lambda+n)}{2n}f(\xi_n,0)+(m-\lambda)\xi_n\frac{\partial f}{\partial\xi_n}(\xi_{n,}0)=0\ \text{ or }\ f(\xi,0)=0,\quad \theta(\xi_{n,}0)\ =\ 1,$$

$$f'(\xi_{n,}\infty)=\xi_n\ ,\quad \theta(\xi_{n,}\infty)=0 \quad (16)$$

and the primes in equations (14)-(16) denote partial differentiations with respect to η.

Note that the ξ_n parameter here represents the forced flow effect on free convection. The case of $\xi_n=0$ corresponds to pure free convection and the limiting case of $\xi_n=\infty$ corresponds to pure forced convection. The above system of equations (14)-(16) was solved over the region covered by $\xi_n=0-1$ to provide the other half of the solution for the entire mixed convection regime.

The velocity components u and v, the local friction factor and the local Nusselt number for this case have the following expressions

$$u\ =\alpha C\ \ x^{\frac{\lambda}{n}}\ f \quad (17)$$

$$v\ =\ -\alpha C^{\frac{1}{2}}\ x^{\frac{\lambda-n}{2n}}\left\{\frac{\lambda+n}{2n}\ f+(m-\lambda)\xi\frac{\partial f}{\partial\xi}+\frac{\lambda-n}{2n}\cdot\eta f'\right\} \quad (18)$$

$$Nu_x=-Ra_x^{\frac{1}{2}}\,\theta'(\xi_{n,}\ 0) \quad (19)$$

Numerical scheme

The numerical scheme to solve equations (6) and (7) adopted here is based on a combination of the following concepts:

(a) The boundary conditions for $\eta = \infty$ are replaced by

$$f'(\xi, \eta_{\max}) = 1, \quad \theta(\xi, \eta_{\max}) = 0 \tag{20}$$

where $\eta_{\max}$ is a sufficiently large value of η at which the boundary conditions (8) are satisfied. $\eta_{\max}$ varies with the value of n. In the present work, a value of $\eta_{\max} = 25$ was checked to be sufficient for free stream behavior.

(b) The two-dimensional domain of interest (ξ, η) is discretized with an equispaced mesh in the ξ-direction and another equispaced mesh in the η-direction.

(c) The partial derivatives with respect to η are evaluated by the second order difference approximation.

(d) Two iteration loops based on the successive substitution are used because of the nonlinearity of the equations.

(e) In each inner iteration loop, the value of ξ is fixed while each of the equations (6) and (7) is solved as a linear second order boundary value problem of ODE on the η-domain. The inner iteration is continued until the nonlinear solution converges with a convergence criterion of 10^{-6} in all cases for the fixed value of ξ.

(f) In the outer iteration loop, the value of ξ is advanced. The derivatives with respect to ξ are updated after every outer iteration step.

In the inner iteration step, the finite difference approximation for equations (6) and (7) is solved as a boundary value problem. The numerical results are affected by the number of mesh points in both directions. To obtain accurate results, a mesh sensitivity study was performed. After some trials, in the η-direction 190 mesh points were chosen whereas in the ξ-direction 41 mesh points were used. The tolerance for convergence was 10^{-6}. Increasing the mesh points to a larger value led to identical results.

Results and discussion

Numerical results for $\theta'(\xi, 0)$ are tabulated in Tables I-III. In order to assess the accuracy of the numerical results, we compare our results for Newtonian fluid

Table Ia.
Comparison of values of $\theta'(\xi_F, 0)$ for n = 1, m = 0 and $\xi_F = 0$

λ	$\theta'(\xi_F, 0)$ Present results	Hsieh *et al.*
0.0	0.56414	0.5642
0.5	0.88602	0.8862
1.0	1.12812	1.1284

(n = 1) with those of Hsieh *et al.* (1993, pp. 1485-93). The agreement between the two is within 0.01 percent difference. Therefore, the present results are highly accurate.

The velocity and temperature profiles are displayed in Figure 2 for prescribed values of m, n, λ and ξ_f. The thermal boundary layer thicknesses decrease as ξ_f increases. The slip velocity at the porous surface $f'(\xi, 0)$ decreases as the viscosity index n increases. The surface temperature gradient and hence the heat transfer rate increases as ξ_f increases. The slip velocity at the wall increases as ξ_f increases. Figure 3 displays the variation of Nusselt number with ξ_f for n ranging from 0.5-1.5. It is observed that the solutions for the forced convection dominated regime and the free

λ	$\theta'(\xi_N, o)$ Present results	Hsieh *et al.*
0.0	0.44362	0.4438
0.5	0.76999	0.7704
1.0	0.99999	1.0000

Table Ib.
Comparison of values of $\theta'(\xi_N, 0)$ for n = 1, m = 0 and $\xi_N = 0$

ξ_F	$-\theta'(\xi_F, 0)$ n = 0.5	n = 1.0	n = 1.5
0.0	0.94151	0.94151	0.94151
0.1	1.01001	0.97336	0.96135
0.2	1.07876	1.00412	0.98033
0.3	1.14770	1.03391	0.99854
0.4	1.21681	1.06281	1.01605
0.5	1.28607	1.09091	1.03294
0.6	1.35547	1.11826	1.04926
0.7	1.42497	1.14492	1.06505
0.8	1.49457	1.17095	1.08036
0.9	1.56426	1.19639	1.09522
1.0	1.63403	1.22127	1.10966
ξ_N	$-\theta'(\xi_N, 0)$ n = 0.5	n = 1.0	n = 1.5
1.0	1.63402	1.22119	1.10967
0.9	1.58273	1.17753	1.08648
0.8	1.49572	1.14247	1.06224
0.7	1.44301	1.10557	1.03680
0.6	1.39554	1.06528	1.01001
0.5	1.35444	1.02270	0.98165
0.4	1.32004	0.97810	0.95149
0.3	1.29278	0.93134	0.91918
0.2	1.27307	0.88220	0.88429
0.1	1.26117	0.83043	0.84618
0.0	1.25718	0.77584	0.80413

Table II.
Values of $-\theta'(\xi_F, 0)$ and $-\theta'(\xi_N, 0)$ for λ = 0.5 and m = 1/3

convection dominated regime meet and match over the mixed convection regime. As λ and ξ_f increase, the Nusselt number increases for a given n. As n increases, the heat transfer rate parameter decreases. As shown in Table III, the Nusselt number increases as the wedge angle parameter m increases.

Concluding remarks

In this paper, we have presented a boundary layer analysis for the mixed convection in non-Newtonian fluids along a vertical wedge embedded in fluid-saturated porous medium. The flow regime was divided into forced convection dominated and natural convection dominated regions. In the forced convection dominated region, $\xi_f = (\frac{Ra_x}{Pe_x})^n$ characterizes the buoyancy effect on forced convection whereas $\xi_n = (\frac{Pe_x}{Pa_x})^n$ is a measure of the effect of forced flow on free convection. Numerical solutions using a finite difference scheme were obtained for the flow and temperature fields for several values of the wedge angle parameter, m, the exponent λ for the surface temperature variation and the viscosity index, n.

	$-\theta'(\xi_F, 0)$			
ξ_F	m = 0	m = 1/3	m = 1/2	m = 1
0.0	0.88602	0.94151	0.96833	1.04522
0.1	0.95462	1.01001	1.03667	1.11274
0.2	1.02342	1.07876	1.10527	1.18062
0.3	1.09239	1.14770	1.17410	1.24878
0.4	1.16151	1.21681	1.24311	1.31719
0.5	1.23076	1.28607	1.31227	1.38580
0.6	1.30014	1.35547	1.38158	1.45457
0.7	1.36962	1.42497	1.45100	1.52349
0.8	1.43921	1.49457	1.52052	1.59253
0.9	1.50889	1.56426	1.59013	1.66166
1.0	1.57866	1.63403	1.65982	1.73088
	$-\theta'(\xi_N, 0)$			
ξ_N	m = 0	m = 1/3	m = 1/2	m = 1
1.0	1.57864	1.63402	1.65980	1.73088
0.9	1.57745	1.58273	1.58723	1.62745
0.8	1.48216	1.49572	1.47415	1.52407
0.7	1.42553	1.44301	1.37248	1.42077
0.6	1.37752	1.39554	1.27495	1.31761
0.5	1.33852	1.35444	1.17892	1.21468
0.4	1.30811	1.32004	1.08359	1.11220
0.3	1.28575	1.29278	0.98909	1.01054
0.2	1.27062	1.27307	0.89600	0.91034
0.1	1.26190	1.26117	0.80525	0.81244
0.0	1.25904	1.25718	0.71651	0.71598

Table III. Values of $-\theta'(\xi_F, 0)$ and $-\theta'(\xi_N, 0)$ for n = 0.5 and λ = 0.5

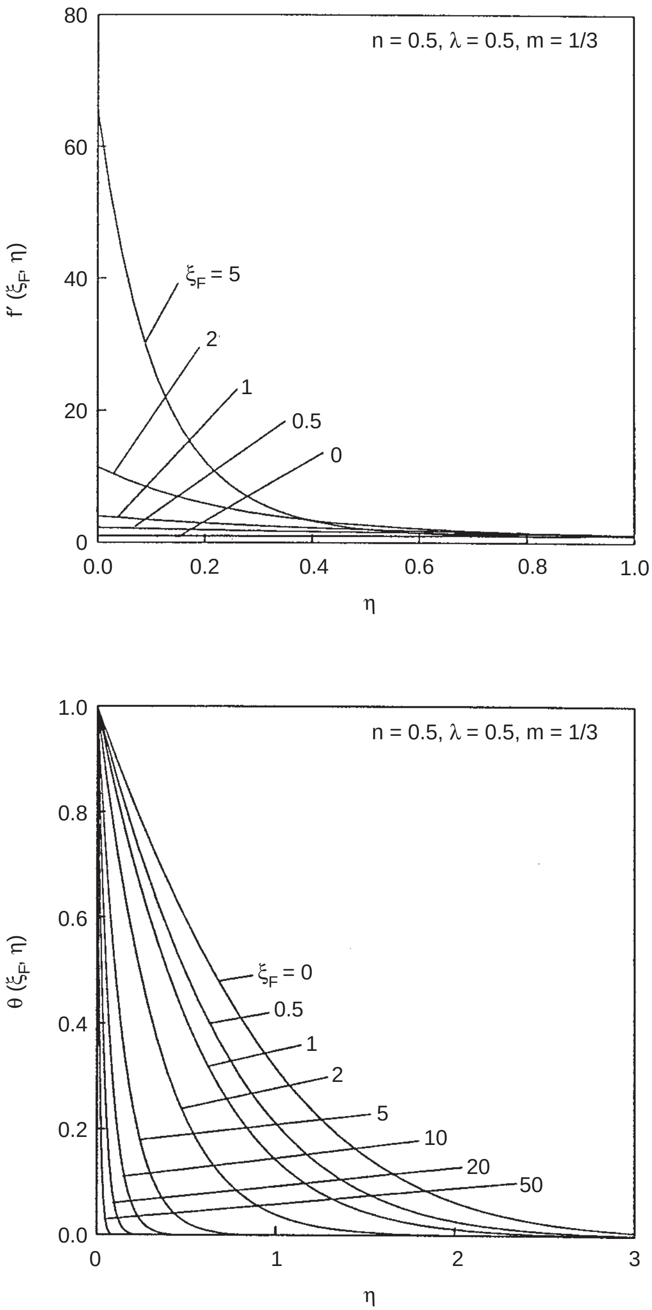

Figure 2.
Velocity and temperature profiles for $n = 0.5$, $\lambda = 0.5$ and $m = 1/3$

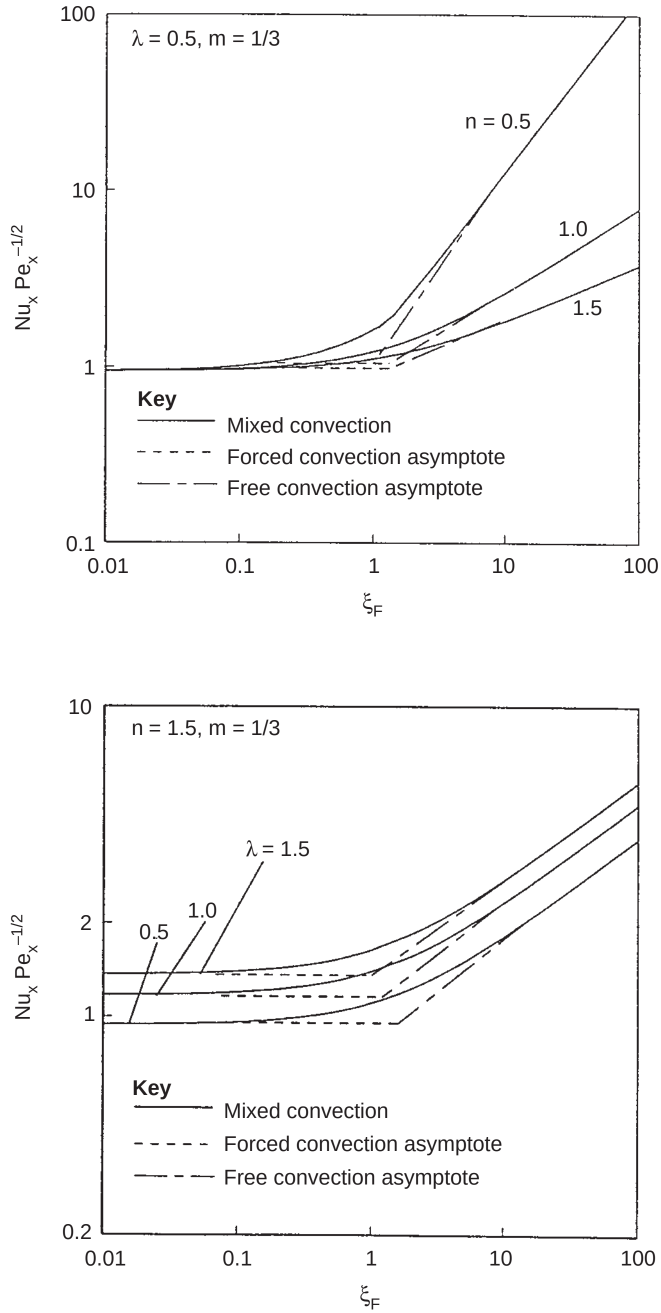

Figure 3.
Local Nusselt number

References

Chen, H.T. and Chen, C.K. (1988), "Free convection of non-Newtonian fluids along a vertical plate embedded in a porous medium", *Transactions of ASME, Journal of Heat Transfer*, Vol. 110, pp. 257-60.

Cheng, P. and Minkowycz, W.J. (1977), "Free convection about a vertical flat plate embedded in a porous medium with application to heat transfer from a dike", *Journal of Geophysical Research*, Vol. 82, pp. 2040-9.

Gorla, R.S.R. and Kumari, M. (1996), "Mixed convection in non-Newtonian fluids along a vertical plate in a porous medium", *Acta Mechanica Journal*, Vol. 118, pp. 55-64.

Gorla, R.S.R. and Kumari, M. (1999), "Combined convection from horizontal surfaces in a non-Newtonian fluid-saturated porous medium", *Journal of Theoretical and Applied Fluid Mechanics* (in print).

Gorla, R.S.R. and Takhar, H.S. (1997), "Mixed convection in non-Newtonian fluids along a vertical plate in porous media with surface mass transfer", *International Journal of Numerical Methods for Heat & Fluid Flow*, Vol. 7, pp. 596-608.

Gorla R.S.R. and Tornabene, R. (1988), "Free convection from a vertical plate with nonuniform surface heat flux and embedded in a porous medium", *Transport in Porous Media Journal*, Vol. 3, pp. 95-106.

Gorla, R.S.R. and Zinolabedini, A. (1987), "Free convection from a vertical plate with nonuniform surface temperature and embedded in a porous medium", *Transactions of ASME, Journal of Energy Resources Technology*, Vol. 109, pp. 26-30.

Gorla, R.S.R., Shanmugam, K. and Kumari, M. (1997a), "Nonsimilar solutions for mixed convection in non-Newtonian fluids along horizontal surfaces in porous media", *Transport in Porous Media Journal*, Vol. 28, pp. 319-34.

Gorla, R.S.R., Shanmugam, K. and Kumari, M. (1997b), "Mixed convection in non-Newtonian fluids along nonisothermal horizontal surfaces in porous media", *Heat and Mass Transfer Journal*, Vol. 33, pp. 281-6.

Hsieh, J.C., Chen, T.S. and Armaly, B.F. (1993), "Nonsimilarity solutions for mixed convection from vertical surfaces in porous media", *International Heat and Mass Transfer Journal*, Vol. 36, pp. 1485-93.

Kumari, M. and Gorla, R.S.R. (1996), "Combined convection in non-Newtonian fluids along a nonisothermal vertical plate in a porous medium", *Transport in Porous Media Journal*, Vol. 24, pp. 157-66.

Kumari, M. and Gorla, R.S.R. (1997), "Combined convection along a non-isothermal wedge in a porous medium", *Heat and Mass Transfer Journal*, Vol. 32, pp. 393-8.

Kumari, M., Gorla, R.S.R. and Byrd, L. (1997), "Mixed convection in non-Newtonian fluids along a horizontal plate in a porous medium", *Trans. ASME, Journal of Energy Resources Technology*, Vol. 119, pp. 34-7.

Minkowycz, W.J., Cheng, P. and Chang, C.H. (1985), "Mixed convection about a nonisothermal cylinder and sphere in a porous medium", *Numerical Heat Transfer*, Vol. 8, pp. 349-59.

Nakayama, A. and Koyama, H. (1991), "Buoyancy-induced flow of non-Newtonian fluids over a non-isothermal body of arbitrary shape in a fluid-saturated porous medium", *Applied Scientific Research*, Vol. 48, pp. 55-70.

Nakayama, A, and Pop, I. (1985), "A unified similarity transformation for free, forced and mixed convection in Darcy and non-Darcy porous media", *International Heat and Mass Transfer Journal*, Vol. 28, pp. 683-97.

Ranganathan, P. and Viskanta, R. (1984), "Mixed convection boundary layer flow along a vertical surface in a porous medium", *Numerical Heat Transfer*, Vol. 7, pp. 305-17.